Continuity and Differentiability

✓ **Continuity :** Suppose f is a real function on a subset of the real numbers and let c be a point in the domain f. Then f is continuous at c if $\boxed{\lim_{x \to c} f(x) = f(c)}$

✓ **Discontinuity :** A function said to be discontinuous at point $x = a$, if it is not continuous at this point. This point $x = a$ where the function is not continuous is called the point of discontinuity.

✓ **Theorem 1** Suppose f and g be two real functions continuous at a real no. then,

(1) $f+g$ is continuous at $x=c$
(2) $f-g$ is continuous at $x=c$
(3) $f \cdot g$ is continuous at $x=c$
(4) $\left(\frac{f}{g}\right)$ is continuous at $x=c$, $\{$provided $g(c) \neq 0\}$

✓ **Theorem 2** Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c.

✓ **Differentiability :** Suppose f is a real function and c is a point in its domain. The derivative of f at c defined by $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}[f(x)]_c$. The function defined by

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ wherever the limit exists is defined to be the derivative of f. The derivative denoted by $f'(x)$ or $\frac{d}{dx}[f(x)]$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y'.

✓ **Algebra of derivaties :**

(i) $\boxed{(u \pm v)' = u' \pm v'}$

(ii) $\boxed{(uv)' = u'v + uv'}$ (Leibnitz or product rule)

(iii) $\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, \text{ wherever } v \neq 0}$ (Quotient Rule)

✓ **Theorem 3.** If a function f is differentiable at a point c, then it is also continuous at that point.

Note : Every differentiable function is continuous.

✓ **Chain Rule :** Let f be a real valued function of which is a composite of two functions u and v i.e. $f = v \circ u$; Suppose $t = u(x)$ and if $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

Suppose f is real valued function which is a composite of three functions u, v and w; i.e. $f = (w \circ u) \circ v$ and if $t = v(x)$ and $s = u(t)$ then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

✓ **Some properties of Logorithmic function**

$\boxed{\log_a p = \frac{\log_b p}{\log_b a}}$ $\boxed{\log_b pq = \log_b p + \log_b q}$ $\boxed{\log_b p^2 = \log_b p + \log_b p = 2 \log p}$

$\boxed{\log_b p^n = n \log p}$ $\boxed{\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y}$ $\boxed{\log_b x = \frac{1}{\log_x b}}$

Note :

Exponential form	logarithim form
$2^3 = 8$	$\log_2 8 = 3$
$b^1 = b$	$\log_b b = 1$
$b^0 = 1$	$\log_b 1 = 0$

Some standard derivative

$\frac{d}{dx}(c) = 0$ c = constant

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\log_e x) = \frac{1}{x}$

$\frac{d}{dx}(a^x) = a^x \log_e a$

$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$

logarithmic differentiation

$y = f(x) = [u(x)]^{v(x)}$

taking log both sides,

$\log y = v(x)\log[u(x)]$

using chain rule to differentiate

$\frac{1}{y}\cdot\frac{dy}{dx} = v(x)\cdot\frac{1}{u(x)}\cdot u'(x) + v'(x)\cdot\log[u(x)]$

$$\boxed{\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\cdot u'(x) + v'(x)\cdot\log[u(x)]\right]}$$

Derivative of functions in Parametric forms

$x = f(t) = g(t)$ parametric form with t as a parameter.

$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ [whenever $\frac{dx}{dt} \neq 0$]

$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ [as $\frac{dy}{dt} = g'(t)$ and $\frac{dx}{dt} = f'(t)$] [provided $f'(t) \neq 0$]

Second order derivative

Let $y = f(x)$

$\frac{dy}{dx} = f'(x)$ ——— (i)

differentiate (i) again w.r.t to x,

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f'(x)] \Rightarrow \boxed{\frac{d^2y}{dx^2} = f''(x)}$ Denoted D^2y or y''

Note : Higher order derivative may be defined similarly

Rolle's Theorem

If $f:[a\ b] \to R$ is continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) = f(b)$ then there exists some c in (a,b) such that $f'(c) = 0$

Langrange Theorem or Mean value theorem

If $f:[a,b] \to R$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists some c in (a,b) such that

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$